Determinantal process starting from an orthogonal symmetry is a Pfaffian process

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Abstract

When the number of particles N is finite, the noncolliding Brownian motion (BM) and the noncolliding squared Bessel process with index $\nu > -1$ (BESQ$^{(\nu)}$) are determinantal processes for arbitrary fixed initial configurations. In the present paper we prove that, if initial configurations are distributed with orthogonal symmetry, they are Pfaffian processes in the sense that any multitime correlation functions are expressed by Pfaffians. The 2×2 skew-symmetric matrix-valued correlation kernels of the Pfaffians processes are explicitly obtained by the equivalence between the noncolliding BM and an appropriate dilatation of a time reversal of the temporally inhomogeneous version of noncolliding BM with finite duration in which all particles start from the origin, $N\delta_0$, and by the equivalence between the noncolliding BESQ$^{(\nu)}$ and that of the noncolliding squared generalized meander starting from $N\delta_0$.

Keywords Determinantal and Pfaffian processes, Eigenvalue distributions of random matrices, Noncolliding Brownian motion, Noncolliding squared Bessel process and generalized meander

1 Introduction

We consider one-dimensional particle systems called *the noncolliding Brownian motion* and *the noncolliding squared Bessel process*. The former is equivalent with Dyson's Brownian motion (BM) model with parameter $\beta = 2$, which was introduced as the eigenvalue process of the Hermitian-matrix-valued BM [9] corresponding to the Gaussian unitary ensemble (GUE) of random matrices [32, 11]. The latter is a one-parameter family indexed by $\nu > -1$ and abbreviated as noncolliding BESQ$^{(\nu)}$ [22]. When the number of particles is finite, $N \in \mathbb{N} \equiv \{1, 2, 3, \dots\}$ and $\nu \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, the noncolliding BESQ$^{(\nu)}$ realizes the eigenvalue process of the matrix-valued diffusion process called the Laguerre process (or complex Wishart process) [26], whose distribution at each time describes squares of singular

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values of $(N+\nu) \times N$ random matrices in the chiral Gaussian unitary ensemble (chGUE) [50, 49], and when $\nu = 1/2$ (resp. $\nu = -1/2$), it expresses the stochastic evolution [17] of the squares of positive eigenvalues of $2N \times 2N$ random matrices in the Gaussian ensemble of class C (resp. class D) [1, 2]. (See [12, 6, 25, 34, 18, 48, 45, 27, 4, 42, 40, 7, 28, 43] for related interacting particle systems.)

In the previous papers [20, 21, 22], it was proved that if the number of particles is finite, $N \in \mathbb{N}$, these two processes are *determinantal* for arbitrary initial configurations $\xi(\cdot) = \sum_{j=1}^{N} \delta_{x_j}(\cdot)$ with $x_1 \leq x_2 \leq \cdots \leq x_N, x_j \in \Lambda, 1 \leq j \leq N$, where $\delta_y(\cdot)$ denotes the delta measure on y; $\delta_y(x) = \delta_{xy}$, $\Lambda = \mathbb{R}$ for the noncolliding BM and $\Lambda = \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$ for the noncolliding BESQ$^{(\nu)}$. Here, given a fixed initial configuration ξ, a process is said to be determinantal, if there is a function $\mathbb{K}(s,x;t,y)$ such that it is continuous with respect to (x,y) for any fixed $(s,t) \in [0,\infty)^2$ and any multitime correlation function is given by a determinant in the form

$$\rho^{\xi}(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}) = \det_{\substack{1 \leq j \leq N_m, 1 \leq k \leq N_n \\ 1 \leq m,n \leq M}} [\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})], \tag{1.1}$$

$M \in \mathbb{N}, 0 < t_1 < \cdots < t_M < \infty, 1 \leq N_m \leq N, 1 \leq m \leq M$, where $\boldsymbol{x}_{N_m}^{(m)} = (x_1^{(m)}, \dots, x_{N_m}^{(m)})$ denotes the points at which observation is performed at time t_m, $1 \leq m \leq M$ [19]. The function $\mathbb{K}$ is called the *correlation kernel* and it determines finite dimensional distributions of the process through (1.1). For a configuration $\xi(\cdot) = \sum_{j=1}^{N} \delta_{x_j}(\cdot)$, shift by $w \in \mathbb{C}$ is denoted by $\tau_w \xi(\cdot) = \sum_{j=1}^{N} \delta_{x_j+w}(\cdot)$ and dilatation by factor $c > 0$ is denoted by $c \circ \xi(\cdot) = \sum_{j=1}^{N} \delta_{cx_j}(\cdot)$. The correlation kernels for the noncolliding BM and the noncolliding BESQ$^{(\nu)}$ are respectively given by

$$\begin{aligned}
\mathbb{K}^{\xi}(s,x;t,y) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R} \setminus [-\varepsilon,\varepsilon]} du \oint_{\mathrm{C}_{iu}(\xi)} dz\, p(s,x|z) \frac{\Pi_{\tau_{-z}\xi}(iu-z)}{iu-z} p(-t,iu|y) \\
&\quad -\mathbf{1}(s>t) p(s-t,x|y), \quad (x,y) \in \mathbb{R}^2, (s,t) \in [0,\infty)^2,
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
\mathbb{K}_{\nu}^{\xi}(s,x;t,y) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{-\varepsilon} du \oint_{\mathrm{C}_{u}(\xi)} dz\, p^{(\nu)}(s,x|z) \frac{\Pi_{\tau_{-z}\xi}(u-z)}{u-z} p^{(\nu)}(-t,u|y) \\
&\quad -\mathbf{1}(s>t) p^{(\nu)}(s-t,x|y), \quad (x,y) \in (0,\infty)^2, (s,t) \in [0,\infty)^2, \nu > -1,
\end{aligned} \tag{1.3}$$

where $i = \sqrt{-1}$, $\mathrm{C}_{z'}(\xi)$ denotes a closed contour on the complex plane $\mathbb{C}$ encircling the points in supp $\xi \equiv \{x \in \Lambda : \xi(\{x\}) > 0\}$ once in the positive direction but not the point z', p and $p^{(\nu)}$ are the extended versions of transition probability densities of the one-dimensional standard BM [21] and the BESQ$^{(\nu)}$ [22],

$$p(t,y|x) = \begin{cases} \dfrac{1}{\sqrt{2\pi|t|}} e^{-(x-y)^2/2t}, & t \in \mathbb{R} \setminus \{0\}, x,y \in \mathbb{C}, \\ \delta(y-x), & t = 0, x,y \in \mathbb{C}, \end{cases} \tag{1.4}$$

$$p^{(\nu)}(t,y|x) = \begin{cases} \dfrac{1}{2|t|}\left(\dfrac{y}{x}\right)^{\nu/2} e^{-(x+y)/2t} I_\nu\left(\dfrac{\sqrt{xy}}{t}\right), & t \in \mathbb{R}\setminus\{0\}, x \in \mathbb{C}\setminus\{0\}, y \in \mathbb{C}, \\ \dfrac{y^\nu e^{-y/2t}}{(2|t|)^{\nu+1}\Gamma(\nu+1)}, & t \in \mathbb{R}\setminus\{0\}, x=0, y \in \mathbb{C}, \\ \delta(y-x), & t=0, x, y \in \mathbb{C}, \end{cases} \tag{1.5}$$

with the Gamma function $\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du$ and the modified Bessel function $I_\nu(z) = \sum_{n=0}^\infty (z/2)^{2n+\nu}/\{\Gamma(n+1)\Gamma(n+1+\nu)\}$, Π_ξ is an entire function having supp ξ as the zero set expressed by the following Weierstrass canonical product with genus 0 [29, 39],

$$\Pi_\xi(z) = \prod_{x\in\xi}\left(1-\frac{z}{x}\right) \equiv \prod_{x\in \text{supp }\xi}\left(1-\frac{z}{x}\right)^{\xi(\{x\})}, \quad z \in \mathbb{C}, \tag{1.6}$$

and $\mathbf{1}(\omega)$ is the indicator of a condition ω; $\mathbf{1}(\omega)=1$ if ω is satisfied and $\mathbf{1}(\omega)=0$ otherwise. In (1.5) we have defined z^ν to be $\exp(\nu \log z)$, where the argument of z is given its principal value; $z^\nu = \exp[\nu\{\log|z| + \sqrt{-1}\arg(z)\}], -\pi < \arg(z) \leq \pi$. We say that the correlation kernels, which are asymmetric, $\mathbb{K}(s,x;t,y) \neq \mathbb{K}(t,y,s,x)$ for $s \neq t$, as (1.2) and (1.3), are *of Eynard-Mehta type* [10, 36, 5, 19, 24].

For $N \in \mathbb{N}, \xi = \sum_{j=1}^N \delta_{x_j}, \boldsymbol{x} = (x_1,\dots,x_N), \beta \geq 1, a > -1, \sigma^2 > 0$, let

$$\mu_{N,\sigma^2}^{(\beta)}(\xi) = \frac{\sigma^{-N\{\beta(N-1)+2\}/2}}{C_N^{(\beta)}} e^{-|\boldsymbol{x}|^2/2\sigma^2}|h_N(\boldsymbol{x})|^\beta, \tag{1.7}$$

$$\mu_{N,\sigma^2}^{(\beta,a)}(\xi) = \frac{\sigma^{-N\{\beta(N-1)+2(a+1)\}}}{C_N^{(\beta,a)}} \prod_{j=1}^N (x_j^a e^{-x_j/2\sigma^2})|h_N(\boldsymbol{x})|^\beta, \tag{1.8}$$

where

$$h_N(\boldsymbol{x}) = \prod_{1\leq j<k\leq N}(x_k - x_j) = \det_{1\leq j,k\leq N}[x_j^{k-1}],$$

$|\boldsymbol{x}|^2 = \sum_{j=1}^N x_j^2$, and the normalization factors are given by

$$C_N^{(\beta)} = \frac{(2\pi)^{N/2}}{N!}\prod_{j=1}^N \frac{\Gamma(j\beta/2+1)}{\Gamma(\beta/2+1)},$$

$$C_N^{(\beta,a)} = \frac{2^{N\{\beta(N-1)+2(a+1)\}/2}}{N!}\prod_{j=1}^N \frac{\Gamma(j\beta/2+1)\Gamma(j\beta/2+a-\beta/2+1)}{\Gamma(\beta/2+1)}.$$

They are the probability density functions of random configurations $\Xi = \sum_{j=1}^N \delta_{X_j}$ and $\widetilde{\Xi} = \sum_{j=1}^N \delta_{\widetilde{X}_j}$ in which the configuration spaces of particle positions $\boldsymbol{X} = (X_1,\dots,X_N)$ and $\widetilde{\boldsymbol{X}} = (\widetilde{X}_1,\dots,\widetilde{X}_N)$ are given by

$$\boldsymbol{X} \in \mathbb{W}_N^{\mathrm{A}} \equiv \{\boldsymbol{x} = (x_1,\dots,x_N) : x_1 < x_2 < \cdots < x_N\},$$
$$\widetilde{\boldsymbol{X}} \in \mathbb{W}_N^{+} \equiv \{\boldsymbol{x} = (x_1,\dots,x_N) : 0 \leq x_1 < x_2 < \cdots < x_N\}.$$

In particular, when $\beta = 1, 2$ and 4, (1.7) gives the distributions of eigenvalues of $N \times N$ Hermitian random matrices in the Gaussian orthogonal ensemble (GOE), GUE, and the Gaussian symplectic ensemble (GSE) with variances σ^2, respectively [32, 11]. Similarly, for $\nu \in \mathbb{N}_0$, (1.8) with $(\beta, a) = (1, (\nu-1)/2), (2, \nu), (4, 2\nu+1)$ give the distributions of squares of (distinct) singular values of $(N+\nu) \times N$ random matrices in the chiral Gaussian orthogonal ensemble (chGOE), chGUE, and the chiral Gaussian symplectic ensemble (chGSE), respectively [50, 49, 46]. Moreover [18], (i) (1.8) with $(\beta, a) = (1, \nu/2)$ was called 'the Laguerre ensemble $\beta = 1$ initial condition' in [12], (ii) (1.8) with $(\beta, a) = (1, 0)$ gives the distribution of squares of positive eigenvalues of $2N \times 2N$ random matrices in the Gaussian ensemble of class CI studied by Altland and Zirnbauer [1, 2], (iii) (1.8) with $(\beta, a) = (1, -1/2)$ gives the distribution of squares of positive eigenvalues of $2N \times 2N$ random matrices in the Gaussian ensemble of 'the real-component version of class D' of the Bogoliubov-de Gennes universality class [17], and (iv) (1.8) with $(\beta, a) = (4, 0)$ and $(4, 2)$ give the distributions of squares of distinct eigenvalues of $2N \times 2N$ random matrices in the Gaussian ensembles of class DIII-even [1, 2] and of class DIII-odd [13], respectively. We write the expectations of measurable functions of Ξ and $\widetilde{\Xi}$, $F(\Xi)$ and $\widetilde{F}(\widetilde{\Xi})$, with respect to distributions (1.7) and (1.8) as $\mathbf{E}^{(\beta)}_{N,\sigma^2}[F(\Xi)]$ and $\mathbf{E}^{(\beta,a)}_{N,\sigma^2}[\widetilde{F}(\widetilde{\Xi})]$, respectively. We can say that [12] the distributions with $\beta = 2$ have *unitary symmetry* and those with $\beta = 1$ do *orthogonal symmetry*.

Recently, we studied the noncolliding BM and the noncolliding BESQ$^{(\nu)}$, $\nu > -1$, starting not from any fixed configurations but from the distributions having unitary symmetry, $\mu^{(2)}_{N,\sigma^2}$ and $\mu^{(2,\nu)}_{N,\sigma^2}$, respectively. We showed that in these cases the determinantal structures of multitime correlation functions are maintained but the correlation kernels are replaced by the time shift $t \to t + \sigma^2$ of the correlation kernels for the special initial configuration $\xi = N\delta_0$, *i.e.*, the configuration in which all N particles are put on the origin [14]. That is, the equalities

$$\begin{aligned}
\rho^{\mu^{(2)}_{N,\sigma^2}}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}) &\equiv \mathbf{E}^{(2)}_{N,\sigma^2}\left[\rho^{\Xi}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M})\right] \\
&= \det_{\substack{1 \le j \le N_m, 1 \le k \le N_n \\ 1 \le m,n \le M}} [\mathbb{K}^{N\delta_0}(t_m + \sigma^2, x^{(m)}_j; t_n + \sigma^2, x^{(n)}_k)], \\
\rho_\nu^{\mu^{(2,\nu)}_{N,\sigma^2}}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}) &\equiv \mathbf{E}^{(2,\nu)}_{N,\sigma^2}\left[\rho_\nu^{\Xi}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M})\right] \\
&= \det_{\substack{1 \le j \le N_m, 1 \le k \le N_n \\ 1 \le m,n \le M}} [\mathbb{K}_\nu^{N\delta_0}(t_m + \sigma^2, x^{(m)}_j; t_n + \sigma^2, x^{(n)}_k)], \quad \nu > -1, \qquad (1.9)
\end{aligned}$$

hold for any $M \in \mathbb{N}$, $0 < t_1 < \cdots < t_M < \infty$, $\sigma^2 > 0$, $\boldsymbol{x}^{(m)}_{N_m} \in \mathbb{W}^{\mathrm{A}}_{N_m}$ or $\boldsymbol{x}^{(m)}_{N_m} \in \mathbb{W}^{+}_{N_m}$, $1 \le N_m \le N, 1 \le m \le M$. We should note that $\mathbb{K}^{N\delta_0}$ and $\mathbb{K}_\nu^{N\delta_0}$ are the correlation kernels known as the *extended Hermite kernel* and *extended Laguerre kernel*, respectively [36, 12, 47, 19, 11].

In the present paper, we report the cases when the noncolliding BM and the noncolliding BESQ$^{(\nu)}$ start from the distributions having orthogonal symmetry, $\mu^{(1)}_{N,\sigma^2}$ and $\mu^{(1,a)}_{N,\sigma^2}$, respectively. For $N \in \mathbb{N}$ and a skew-symmetric $2N \times 2N$ matrix $A = (a_{jk})$, the Pfaffian is defined

as

$$\mathrm{Pf}(A) = \mathrm{Pf}_{1\le j<k\le 2N}(a_{jk}) = \frac{1}{N!}{\sum_{\pi}}' \mathrm{sgn}(\pi) a_{\pi(1)\pi(2)} a_{\pi(3)\pi(4)} \cdots a_{\pi(2N-1)\pi(2N)}, \tag{1.10}$$

where the summation ${\sum_{\pi}}'$ is extended over all permutations π of $(1, 2, \dots, 2N)$ with restriction $\pi(2k-1) < \pi(2k), k = 1, 2, \dots, N$. The main result of the present paper is the fact that for any $\sigma^2 > 0$ we can explicitly determine the 2×2 *skew-symmetric matrix-valued correlation kernels*

$$\mathbb{A}(s, x; t, y; \sigma^2) = \begin{pmatrix} \mathrm{A}_{11}(s, x; t, y; \sigma^2) & \mathrm{A}_{12}(s, x; t, y; \sigma^2) \\ -\mathrm{A}_{12}(t, y; s, x; ; \sigma^2) & \mathrm{A}_{22}(s, x; t, y; ; \sigma^2) \end{pmatrix}, \tag{1.11}$$

$(x, y) \in \mathbb{R}^2, (s, t) \in [0, \infty)^2$, and

$$\mathbb{A}^{(\nu,\kappa)}(s, x; t, y; \sigma^2) = \begin{pmatrix} \mathrm{A}_{11}^{(\nu,\kappa)}(s, x; t, y; \sigma^2) & \mathrm{A}_{12}^{(\nu,\kappa)}(s, x; t, y; \sigma^2) \\ -\mathrm{A}_{12}^{(\nu,\kappa)}(t, y; s, x; \sigma^2) & \mathrm{A}_{22}^{(\nu,\kappa)}(s, x; t, y; \sigma^2) \end{pmatrix}, \tag{1.12}$$

$(x.y) \in (0, \infty)^2, (s, t) \in [0, \infty)^2$, with $\kappa = 2(\nu - a)$ such that

$$\begin{aligned} &\rho^{\mu^{(1)}_{N,\sigma^2}}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}) \equiv \mathbf{E}^{(1)}_{N,\sigma^2}\Big[\rho^{\Xi}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M})\Big] \\ &\quad = \mathrm{Pf}_{\substack{1\le j\le N_m, 1\le k\le N_n \\ 1\le m,n\le M}}[\mathbb{A}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2)], \end{aligned} \tag{1.13}$$

$$\begin{aligned} &\rho_{\nu}^{\mu^{(1,a)}_{N,\sigma^2}}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}) \equiv \mathbf{E}^{(1,a)}_{N,\sigma^2}\Big[\rho_{\nu}^{\Xi}(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M})\Big] \\ &\quad = \mathrm{Pf}_{\substack{1\le j\le N_m, 1\le k\le N_n \\ 1\le m,n\le M}}[\mathbb{A}^{(\nu,\kappa)}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2)], \quad \nu > -1, a \in (-1, \nu], \end{aligned} \tag{1.14}$$

hold for any $0 < t_1 < \dots < t_M < \infty$, $\boldsymbol{x}^{(m)}_{N_m} \in \mathbb{W}^{\mathrm{A}}_{N_m}$ or $\boldsymbol{x}^{(m)}_{N_m} \in \mathbb{W}^{+}_{N_m}$, $1 \le N_m \le N, 1 \le m \le M$. As an analogue of a determinantal process, an interacting particle system is said to be a *Pfaffian process*, if any multitime correlation function is given by a Pfaffian [33, 41, 30, 31, 37, 12, 38, 34, 15, 18, 35]. Then we can state that noncolliding diffusion processes, which are determinantal processes if they start from fixed initial configurations, behave as Pfaffian processes when they start from distributions having orthogonal symmetry.

The present paper is organized as follows. In Section 2 preliminaries and main results are given. In Section 3 is devoted to proofs of results.

2 Preliminaries and Main Results

Let $N \in \mathbb{N}, 0 < T < \infty$, and choose an initial configuration $\xi = \sum_{j=1}^N \delta_{x_j}, x_1 \le x_2 \le \dots \le x_N$. Then consider the N-particle system of one-dimensional standard BMs starting from ξ at time $t = 0$ *conditioned never to collide with each other* during time period $(0, T]$ [16]. If the initial configuration is $\xi = N\delta_0$, that is, all N particles start from the origin, we can

show that the multitime joint probability density function for arbitrary $M+1$ sequence of times $0 < t_1 < \cdots < t_M < t_{M+1} \equiv T$, $M \in \mathbb{N}$, is given by the formula

$$p_T^{N\delta_0}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}; t_{M+1}, \xi^{(M+1)}) = C_{N,T}(t_1) \mathrm{sgn}(h_N(\boldsymbol{x}^{(M+1)})) \\ \times \prod_{m=1}^{M} f(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) h_N(\boldsymbol{x}^{(1)}) \prod_{j=1}^{N} p(t_1, x_j^{(1)} | 0), \tag{2.1}$$

where $\xi^{(m)} = \sum_{j=1}^N \delta_{x_j^{(m)}}$, $\boldsymbol{x}^{(m)} = (x_1^{(m)}, \dots, x_N^{(m)}) \in \mathbb{W}_N^{\mathrm{A}}$, $1 \le m \le M+1$,

$$f(t, \boldsymbol{y} | \boldsymbol{x}) = \det_{1 \le j,k \le N} [p(t, y_j | x_k)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}, t \ge 0,$$

and $C_{N,T}(t) = \pi^{N/2} \{\prod_{j=1}^N \Gamma(j/2)\}^{-1} T^{N(N-1)/4} t^{-N(N-1)/2}$ [38, 15]. The process, whose finite dimensional distributions are determined by the formula (2.1), is temporally inhomogeneous [16, 23]. In this paper we call it 'the noncolliding BM with duration T starting from $N\delta_0$' and express it by $(\Xi_T(t), t \in [0, T], \mathbb{P}^{N\delta_0})$.

When we take the limit $T \to \infty$, we have a temporally homogeneous system [16, 19], which we simply call the noncolliding BM (starting from $N\delta_0$). For the noncolliding BM, multitime joint probability density function is given by the following for an arbitrary initial configuration ξ with $\xi(\mathbb{R}) = N \in \mathbb{N}$,

$$p^{\xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}) = h_N(\boldsymbol{x}^{(M)}) \prod_{m=1}^{M-1} f(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) h_N^{(+)}(t_1, \boldsymbol{x}^{(1)}; \xi), \tag{2.2}$$

$0 < t_1 < \cdots < t_M < \infty$, with

$$h_N^{(+)}(t, \boldsymbol{y}; \xi) = \det_{1 \le j,k \le N} \left[\frac{1}{2\pi i} \oint_{\mathrm{C}(\xi_j)} dz \, \frac{p(t, y_k | z)}{\prod_{x \in \xi_j} (z - x)} \right], \quad t \ge 0, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}},$$

where, for a given initial configuration $\xi = \sum_{j=1}^N \delta_{x_j}, x_1 \le x_2 \le \cdots \le x_N$, we define $\xi_j = \sum_{k=1}^j \delta_{x_k}, 1 \le j \le N$, and $\mathrm{C}(\xi)$ denotes a closed contour on the complex plane $\mathbb{C}$ encircling the points in $\mathrm{supp}\, \xi \equiv \{x \in \Lambda : \xi(\{x\}) > 0\}$ once in the positive direction [3, 21]. The noncolliding BM starting from ξ is the temporally homogeneous process, whose finite dimensional distributions are determined by (2.2), and is denoted by $(\Xi(t), t \in [0, \infty), \mathbb{P}^{\xi})$ in this paper. We can prove that $\Xi(t, \cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), t \ge 0$ solves the following system of stochastic differential equations (SDEs),

$$dX_j(t) = dB_j(t) + \sum_{1 \le k \le N, k \ne j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \le j \le N, \quad t \ge 0,$$

with independent one-dimensional standard BMs $\{B_j(t)\}_{j=1}^N$, which is the $\beta = 2$ case of Dyson's BM model [19, 23].

In [18], a temporally inhomogeneous noncolliding diffusion process was introduced, which is called *the noncolliding squared generalized meander* with duration T. It is a two-parameter family of processes indexed by $\nu > -1$ and $\kappa \in [0, 2(\nu+1))$ starting from the configuration $N\delta_0, N \in \mathbb{N}$, which includes the processes studied in [12] and [34] as special cases. This family of processes is denoted here by $(\Xi_T^{(\nu,\kappa)}(t), t \in [0,T], \mathbb{P}^{N\delta_0})$. The multitime joint probability density function is given by

$$
\begin{aligned}
p_{T,(\nu,\kappa)}^{N\delta_0}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}; t_{M+1}, \xi^{(M+1)}) &= C_{N,T}^{(\nu,\kappa)}(t_1) \mathrm{sgn}(h_N(\boldsymbol{x}^{(M+1)})) \\
&\times \prod_{m=1}^{M} f^{(\nu,\kappa)}(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) h_N(\boldsymbol{x}^{(1)}) \prod_{j=1}^{N} p^{(\nu,\kappa)}(t_1, x_j^{(1)} | 0),
\end{aligned}
\tag{2.3}
$$

$0 < t_1 < \cdots < t_M < t_{M+1} \equiv T$, where

$$
p^{(\nu,\kappa)}(t, y|x) = \begin{cases}
\dfrac{1}{2t} \left(\dfrac{y}{x} \right)^{(\nu-\kappa)/2} e^{-(x+y)/2t} I_\nu \left(\dfrac{\sqrt{xy}}{t} \right), & t > 0, x > 0, y \geq 0, \\
\dfrac{y^{\nu-\kappa/2} e^{-y/2t}}{(2t)^{\nu+1} \Gamma(\nu+1)}, & t > 0, x = 0, y \geq 0, \\
\delta(y - x), & t = 0, x, y \geq 0,
\end{cases}
\tag{2.4}
$$

$$
f^{(\nu,\kappa)}(t, \boldsymbol{y} | \boldsymbol{x}) = \det_{1 \leq j,k \leq N} [p^{(\nu,\kappa)}(t, y_j | x_k)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^+, t \geq 0,
$$

and

$$
C_{N,T}^{(\nu,\kappa)}(t) = \frac{T^{(N+\kappa-1)N/2} t^{-(N-1)N}}{2^{N(N-\kappa-1)/2}} \prod_{j=1}^{N} \frac{\Gamma(\nu+1)\Gamma(1/2)}{\Gamma(j/2)\Gamma((j+1+2\nu-\kappa)/2)}.
$$

On the other hand, for the noncolliding BESQ$^{(\nu)}$, which is obtained as temporally homogeneous limit $T \to \infty$ of $\Xi_T^{(\nu,\kappa)}(t)$ [17, 18], the multitime joint probability density function is obtained for an arbitrary initial configuration $\xi = \sum_{j=1}^{N} \delta_{x_j}, 0 \leq x_1 \leq x_2 \leq \cdots \leq x_N, N \in \mathbb{N}$ as

$$
p_\nu^\xi(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}) = h_N(\boldsymbol{x}^{(M)}) \prod_{m=1}^{M-1} f^{(\nu)}(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)} | \boldsymbol{x}^{(m)}) h_N^{(\nu,+)}(t_1, \boldsymbol{x}^{(1)}; \xi), \tag{2.5}
$$

$0 < t_1 < \cdots < t_M < \infty$, with [8, 22]

$$
\begin{aligned}
f^{(\nu)}(t, \boldsymbol{y} | \boldsymbol{x}) &= \det_{1 \leq j,k \leq N} [p^{(\nu)}(t, y_j | x_k)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^+, t \geq 0, \\
h_N^{(\nu,+)}(t, \boldsymbol{y}; \xi) &= \det_{1 \leq j,k \leq N} \left[\frac{1}{2\pi i} \oint_{\mathrm{C}(\xi_j)} dz \, \frac{p^{(\nu)}(t, y_k | z)}{\prod_{x \in \xi_j} (z - x)} \right].
\end{aligned}
$$

We write the noncolliding BESQ$^{(\nu)}$ starting from ξ as $(\Xi^{(\nu)}(t), t \in [0, \infty), \mathbb{P}^\xi)$. If we set

$\Xi^{(\nu)}(t,\cdot) = \sum_{j=1}^{N} \delta_{\widetilde{X}_j(t)}(\cdot)$, $\widetilde{\boldsymbol{X}}(t) = (\widetilde{X}_1(t), \dots, \widetilde{X}_N(t))$ satisfies the SDEs

$$\begin{aligned} d\widetilde{X}_j(t) &= 2\sqrt{\widetilde{X}_j(t)} d\widetilde{B}_j(t) + 2(\nu+1)dt \\ &\quad +4\widetilde{X}_j(t) \sum_{1 \le k \le N, k \neq j} \frac{dt}{\widetilde{X}_j(t) - \widetilde{X}_k(t)}, \quad 1 \le j \le N, \quad t \ge 0, \end{aligned}$$

where $\{\widetilde{B}_j(t)\}_{j=1}^N$ are independent one-dimensional standard BMs and, if $-1 < \nu < 0$, the reflection boundary condition is assumed at the origin [22].

We write the noncolliding BM starting from the GOE eigenvalue distribution $\mu_{N,\sigma^2}^{(1)}$ as $(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}^{(1)}})$ and the noncolliding BESQ$^{(\nu)}, \nu > -1$ from the distribution having orthogonal symmetry, $\mu_{N,\sigma^2}^{(1,a)}, a \in (-1,\nu]$, as $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}^{(1,a)}})$.

In general, two processes having the same state space are said to be *equivalent* if they have the same finite-dimensional distributions, that is, if, for any finite sequence of times $0 < t_1 < \cdots < t_M < \infty, M \in \mathbb{N}$, the multitime joint probability density functions coincide with each other [44]. The key lemma of the present study is the following equivalence.

Lemma 2.1 *For $\sigma^2 > 0$, let*

$$c_{\sigma^2}(t) = \frac{\sigma^2}{\sigma^2 + t}, \quad t \in [0,\infty). \tag{2.6}$$

Then

$$(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}^{(1)}}) = \left(\frac{1}{c_{\sigma^2}(t)} \circ \Xi_{\sigma^2}(\sigma^2 c_{\sigma^2}(t)), t \in [0,\infty), \mathbb{P}^{N\delta_0} \right). \tag{2.7}$$

If the relation

$$a = \nu - \frac{\kappa}{2}, \quad \nu > -1, \quad a \in (-1,\nu], \tag{2.8}$$

is satisfied, then

$$(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}^{\mu_{N,\sigma^2}^{(1,a)}}) = \left(\frac{1}{c_{\sigma^2}(t)^2} \circ \Xi_{\sigma^2}^{(\nu,\kappa)}(\sigma^2 c_{\sigma^2}(t)), t \in [0,\infty), \mathbb{P}^{N\delta_0} \right). \tag{2.9}$$

Remark that

$$0 < s < t < \infty \quad \Longleftrightarrow \quad \sigma^2 > \sigma^2 c_{\sigma^2}(s) > \sigma^2 c_{\sigma^2}(t) > 0. \tag{2.10}$$

Therefore, the RHS of (2.7) and (2.9) are time reverses of the processes $\Xi_T(\cdot)$ and $\Xi_T^{(\nu,\kappa)}(\cdot)$ with duration $T = \sigma^2$, followed by dilatation with factors $1/c_{\sigma^2}(\cdot)$ and $1/c_{\sigma^2}(\cdot)^2$, respectively.

In [38, 15] and [18], it was proved that $(\Xi_T(t), t \in [0,T], \mathbb{P}^{N\delta_0})$ and $(\Xi_T^{(\nu,\kappa)}(t), t \in [0,T], \mathbb{P}^{N\delta_0})$, $0 < T < \infty$, $\nu > -1$, $\kappa \in [0, 2(\nu+1))$ are Pfaffian processes, respectively. Then by the equivalence (2.7) and (2.9) of Lemma 2.1, the following main Theorems are obtained. (For simplicity of expressions, we show the elements of the matrix-valued correlation kernels only

for the case that the number of particles N is even. See, for example, [35] for the general theory of Pfaffian expressions of correlation functions.) Let H_n and L_n^{ν} be the Hermite polynomial of degree n and the Laguerre polynomials of degree n with index ν;

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}, \\ L_n^{\nu}(x) &= \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+\nu+1)x^k}{\Gamma(k+\nu+1)(n-k)!k!}, \quad n \in \mathbb{N}_0, \end{aligned}$$

where $[r]$ denotes the greatest integer not greater than r. For $n \in \mathbb{Z} \equiv \{\dots, -1, 0, 1, 2, \dots\}$ and $\alpha \in \mathbb{R}$ we define

$$\binom{n+\alpha}{n} = \begin{cases} \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}, & \text{if } n \in \mathbb{N},\ \alpha \notin \mathbb{Z}_-, \\ \dfrac{(-1)^n \Gamma(-\alpha)}{\Gamma(n+1)\Gamma(-n-\alpha)}, & \text{if } n \in \mathbb{N},\ n+\alpha \in \mathbb{Z}_-, \\ 0, & \text{if } n \in \mathbb{N},\ \alpha \in \mathbb{Z}_-,\ n+\alpha \in \mathbb{N}_0, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z}_-, \end{cases}$$

where $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}_0$.

Theorem 2.2 *The noncolliding BM with a finite number of particles $N \in \mathbb{N}$, starting from the GOE eigenvalue distribution with variance $\sigma^2 > 0$, $(\Xi(t), t \in [0,\infty), \mathbb{P}^{\mu^{(1)}_{N,\sigma^2}})$, is a Pfaffian process. When N is even, the elements of the matrix-valued correlation kernel (1.11) are given by*

$$\begin{aligned} &A_{11}(s,x;t,y;\sigma^2) = \sum_{k=0}^{N/2-1} \frac{1}{d_k(\sigma^2)} \Big[B_{2k}(s,x;\sigma^2) B_{2k+1}(t,y;\sigma^2) - B_{2k+1}(s,x;\sigma^2) B_{2k}(t,y;\sigma^2) \Big], \\ &A_{12}(s,x;t,y;\sigma^2) \\ &\quad = \begin{cases} \displaystyle\sum_{k=0}^{N/2-1} \frac{1}{d_k(\sigma^2)} \Big[B_{2k+1}(s,x;\sigma^2) C_{2k}(t,y;\sigma^2) - B_{2k}(s,x;\sigma^2) C_{2k+1}(t,y;\sigma^2) \Big], & \text{\textit{if} } s \le t, \\ \displaystyle -\sum_{k=N/2}^{\infty} \frac{1}{d_k(\sigma^2)} \Big[B_{2k+1}(s,x;\sigma^2) C_{2k}(t,y;\sigma^2) - B_{2k}(s,x;\sigma^2) C_{2k+1}(t,y;\sigma^2) \Big], & \text{\textit{if} } s > t, \end{cases} \\ &A_{22}(s,x;t,y;\sigma^2) = \sum_{k=N/2}^{\infty} \frac{1}{d_k(\sigma^2)} \Big[C_{2k}(s,x;\sigma^2) C_{2k+1}(t,y;\sigma^2) - C_{2k+1}(s,x;\sigma^2) C_{2k}(t,y;\sigma^2) \Big], \end{aligned} \tag{2.11}$$

with

$$
\begin{aligned}
d_k(\sigma^2) &= 2\sigma^2\Gamma(k+1/2)\Gamma(k+1), \\
B_{2k}(s,x;\sigma^2) &= \left(\frac{\sigma^2+2s}{4\sigma^2}\right)^k e^{-x^2/2(\sigma^2+s)} H_{2k}\left(\frac{x}{\sqrt{\sigma^2+2s}}\right), \\
B_{2k+1}(s,x;\sigma^2) &= \left(\frac{\sigma^2+2s}{4\sigma^2}\right)^{(2k+1)/2} e^{-x^2/2(\sigma^2+s)} \\
&\times \left\{ H_{2k+1}\left(\frac{x}{\sqrt{\sigma^2+2s}}\right) - \frac{4k\sigma^2}{\sigma^2+2s} H_{2k-1}\left(\frac{x}{\sqrt{\sigma^2+2s}}\right)\right\}, \\
C_{2k}(s,x;\sigma^2) &= \frac{(2k)!}{k!} 2^{-2k+1}\sigma\sqrt{\frac{\sigma^2}{\sigma^2+2s}} e^{x^2/2(\sigma^2+s)-x^2/(\sigma^2+2s)} \\
&\times \sum_{\ell=k}^{\infty} \frac{\ell!}{(2\ell+1)!}\left(\frac{\sigma^2}{\sigma^2+2s}\right)^{(2\ell+1)/2} H_{2\ell+1}\left(\frac{x}{\sqrt{\sigma^2+2s}}\right), \\
C_{2k+1}(s,x;\sigma^2) &= -2^{-2k+1}\sigma\sqrt{\frac{\sigma^2}{\sigma^2+2s}}\left(\frac{\sigma^2}{\sigma^2+2s}\right)^k \\
&\times e^{x^2/2(\sigma^2+s)-x^2/(\sigma^2+2s)} H_{2k}\left(\frac{x}{\sqrt{\sigma^2+2s}}\right), \quad k \in \mathbb{N}_0. \qquad (2.12)
\end{aligned}
$$

Theorem 2.3 *The noncolliding BESQ*$^{(\nu)}, \nu > -1$ *with a finite number of particles* $N \in \mathbb{N}$*, starting from the distribution having orthogonal symmetry,* $\mu^{(1,a)}_{N,\sigma^2}, \sigma^2 > 0, a \in (-1,\nu]$*,* $(\Xi^{(\nu)}(t), t \in [0,\infty), \mathbb{P}^{\mu^{(1,a)}_{N,\sigma^2}})$*, is a Pfaffian process. Let* $\kappa = 2(\nu - a)$*. Then, when* N *is even, the elements of the matrix-valued correlation kernel (1.12) are given by*

$$
\begin{aligned}
&A^{(\nu,\kappa)}_{11}(s,x;t,y;\sigma^2) \\
&\quad = \sum_{k=0}^{N/2-1} \frac{1}{d^{(\nu,\kappa)}_k(\sigma^2)}\Big[B^{(\nu,\kappa)}_{2k}(s,x;\sigma^2)B^{(\nu,\kappa)}_{2k+1}(t,y;\sigma^2) - B^{(\nu,\kappa)}_{2k+1}(s,x;\sigma^2)B^{(\nu,\kappa)}_{2k}(t,y;\sigma^2)\Big], \\
&A^{(\nu,\kappa)}_{12}(s,x;t,y;\sigma^2) \\
&\quad = \sum_{k=0}^{N/2-1} \frac{1}{d_k(\sigma^2)}\Big[B^{(\nu,\kappa)}_{2k+1}(s,x;\sigma^2)C^{(\nu,\kappa)}_{2k}(t,y;\sigma^2) - B^{(\nu,\kappa)}_{2k}(s,x;\sigma^2)C^{(\nu,\kappa)}_{2k+1}(t,y;\sigma^2)\Big] \\
&\qquad -\mathbf{1}(s>t)p^{(\nu,\kappa)}_{-}(s,x;t,y;\sigma^2), \\
&A^{(\nu,\kappa)}_{22}(s,x;t,y;\sigma^2) \\
&\quad = \sum_{k=N/2}^{\infty} \frac{1}{d^{(\nu,\kappa)}_k(\sigma^2)}\Big[C^{(\nu,\kappa)}_{2k}(s,x;\sigma^2)C^{(\nu,\kappa)}_{2k+1}(t,y;\sigma^2) - C^{(\nu,\kappa)}_{2k+1}(s,x;\sigma^2)C^{(\nu,\kappa)}_{2k}(t,y;\sigma^2)\Big],
\end{aligned}
$$

(2.13)

with

$$
\begin{aligned}
d_k^{(\nu,\kappa)}(\sigma^2) &= 2^{-2\nu}\sigma^{-2\kappa}\frac{(2k)!\Gamma(2k+2+2\nu-\kappa)}{\Gamma(\nu+1)^2},\\
B_k^{(\nu,\kappa)}(s,x;\sigma^2) &= \frac{k!}{2^{\nu+1}\Gamma(\nu+1)}\frac{1}{\sigma^{2(\kappa+1)}(\sigma^2+s)^{\nu-\kappa}}\\
&\times\ e^{-x/2(\sigma^2+s)}x^{\nu-\kappa/2}\sum_{j=0}^{k}\alpha_{k,j}\left(\frac{\sigma^2+2s}{\sigma^2}\right)^j L_j^\nu\left(\frac{x}{\sigma^2+2s}\right),\\
C_{2k}^{(\nu,\kappa)}(s,x;\sigma^2) &= \frac{(2k)!\Gamma(2\nu-\kappa+1)}{2^{\nu-1}\Gamma(\nu+1)}\sigma^2\frac{(\sigma^2+s)^{\nu-\kappa}}{(\sigma^2+2s)^{\nu+1}}\binom{2k+2\nu-\kappa+1}{2k+1}\\
&\times\ e^{x/2(\sigma^2+s)-x/(\sigma^2+2s)}x^{\kappa/2}\sum_{j=2k+1}^{\infty}\beta_{j,2k+1}\frac{\Gamma(j+1)}{\Gamma(j+1+\nu)}\left(\frac{\sigma^2}{\sigma^2+2s}\right)^j L_j^\nu\left(\frac{x}{\sigma^2+2s}\right),\\
C_{2k+1}^{(\nu,\kappa)}(s,x;\sigma^2) &= -\frac{(2k+1)!\Gamma(2\nu-\kappa+1)}{2^{\nu-1}\Gamma(\nu+1)}\sigma^2\frac{(\sigma^2+s)^{\nu-\kappa}}{(\sigma^2+2s)^{\nu+1}}\binom{2k+2\nu-\kappa+1}{2k+1}\\
&\times\ e^{x/2(\sigma^2+s)-x/(\sigma^2+2s)}x^{\kappa/2}\sum_{j=2k}^{\infty}\beta_{j,2k}\frac{\Gamma(j+1)}{\Gamma(j+1+\nu)}\left(\frac{\sigma^2}{\sigma^2+2s}\right)^j L_j^\nu\left(\frac{x}{\sigma^2+2s}\right),
\end{aligned}
\tag{2.14}
$$

$k\in\mathbb{N}_0$, *where*

$$
\alpha_{k,j}=\begin{cases}\binom{k-j+\nu-\kappa}{k-j}, & \text{if } k \text{ is even},\\ \frac{k+2\nu-\kappa}{k}\binom{k-2-j+\nu-\kappa}{k-2-j}-\binom{k-j+\nu-\kappa}{k-j}, & \text{if } k \text{ is odd},\end{cases}
$$

$$
\beta_{j,2k}=\binom{j-2k-\nu+\kappa-2}{j-2k},\quad j\geq 2k,
$$

$$
\beta_{j,2k+1}=-\sum_{\ell=k+1}^{[(j+1)/2]} b(2k+3,2\ell-1)\binom{j-2\ell-\nu+\kappa-1}{j-2\ell+1},\quad j\geq 2k+1,
$$

with

$$
b(m,n)=\begin{cases}\prod_{\ell=0}^{(n-m)/2}\frac{m+2\ell+2\nu-\kappa}{m+2\ell}, & \text{if } m,m \text{ are odd and } m\leq n,\\ 1, & \text{if } m,n \text{ are odd and } m>n,\\ 0, & \text{otherwise},\end{cases}
$$

and

$$
p_-^{(\nu,\kappa)}(s,x;t,y;\sigma^2)=\begin{cases}\left(\frac{\sigma^2+s}{\sigma^2+t}\right)^{\nu-\kappa}e^{x/2(\sigma^2+s)-y/2(\sigma^2+t)}p^{(\nu,\kappa)}(s-t,y|x), & x>0,\\ \left(\frac{\sigma^2+t}{\sigma^2}\right)^{-(\nu-\kappa)}\left(\frac{\sigma^2+s}{\sigma^2}\right)^{\kappa}e^{-y/2(\sigma^2+t)}p^{(\nu,\kappa)}(s-t,y|0), & x=0,\end{cases}
$$

for $s > t, y \geq 0$.

3 Proofs of Theorems

3.1 Proof of Lemma 2.1

When the initial configuration $\xi = \sum_{j=1}^{N} \delta_{x_j}$ is distributed according to $\mu^{(1)}_{N,\sigma^2}, \sigma^2 > 0$, there is no multiple point in $\boldsymbol{x} = (x_1, \dots, x_N)$ with probability one, *i.e.*, $\mathbf{P}^{(1)}_{N,\sigma^2}[\boldsymbol{X} \in \mathbb{W}_N^{\rm A}] = 1$. In this case $h_N^{(+)}(t, \boldsymbol{y}; \xi) = f(t, \boldsymbol{y}|\boldsymbol{x})/h_N(\boldsymbol{x})$, and we can confirm the equality

$$\begin{aligned} & h_N(\boldsymbol{x}^{(M)}) h_N^{(+)}(t_1, \boldsymbol{x}^{(1)}; \xi) \mu^{(1)}_{N,\sigma^2}(\xi) \\ & = C_{N,\sigma^2}(\sigma^2 c_{\sigma^2}(t_M)) h_N(c_{\sigma^2}(t_M)\boldsymbol{x}^{(M)}) \prod_{j=1}^{N} p(\sigma^2 c_{\sigma^2}(t_M), c_{\sigma^2}(t_M) x_j^{(M)}|0) \\ & \qquad \times c_{\sigma^2}(t_M)^{N/2} c_{\sigma^2}(t_1)^{N/2} e^{|\boldsymbol{x}^{(M)}|^2/2(\sigma^2+t_M) - |\boldsymbol{x}^{(1)}|^2/2(\sigma^2+t_1)} \\ & \qquad \times f(\sigma^2 - \sigma^2 c_{\sigma^2}(t_1), \boldsymbol{x}|c_{\sigma^2}(t_1)\boldsymbol{x}^{(1)}) \mathrm{sgn}(h_N(\boldsymbol{x})), \end{aligned}$$

where, for $c > 0, \boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{W}_N^{\rm A}$, we put $c\boldsymbol{y} \equiv (cy_1, \dots, cy_N)$. Similarly, we can see the equalities for $1 \leq m \leq M-1$,

$$\begin{aligned} & f(t_{m+1} - t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}) \\ & \quad = c_{\sigma^2}(t_m)^{N/2} c_{\sigma^2}(t_{m+1})^{N/2} e^{|\boldsymbol{x}^{(m)}|^2/2(\sigma^2+t_m) - |\boldsymbol{x}^{(m+1)}|^2/2(\sigma^2+t_{m+1})} \\ & \quad \times f(\sigma^2 c_{\sigma^2}(t_m) - \sigma^2 c_{\sigma^2}(t_{m+1}), c_{\sigma^2}(t_m)\boldsymbol{x}^{(m)}|c_{\sigma^2}(t_{m+1})\boldsymbol{x}^{(m+1)}). \end{aligned}$$

Then for any $M \in \mathbb{N}$, $0 < t_1 < t_2 < \cdots < t_M < \infty$, $\xi = \sum_{j=1}^{N} \delta_{x_j}, \boldsymbol{x} \in \mathbb{W}_N^{\rm A}$, and $\xi^{(m)} = \sum_{j=1}^{N} \delta_{x_j^{(m)}}, \boldsymbol{x}^{(m)} \in \mathbb{W}_N^{\rm A}, 1 \leq m \leq M$, the equality

$$\begin{aligned} & \mu^{(1)}_{N,\sigma^2}(\xi) p^{\xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}) \\ & = p_{\sigma^2}^{N\delta_0}(\sigma^2 c_{\sigma^2}(t_M), c_{\sigma^2}(t_M) \circ \xi^{(M)}; \dots; \sigma^2 c_{\sigma^2}(t_1), c_{\sigma^2}(t_1) \circ \xi^{(1)}; \sigma^2, \boldsymbol{x}) \prod_{m=1}^{M} c_{\sigma^2}(t_m)^N \end{aligned} \tag{3.1}$$

holds. Integration of the LHS of (3.1) over $\boldsymbol{x} \in \mathbb{W}_N^{\rm A}$ gives

$$\begin{aligned} \int_{\mathbb{W}_N^{\rm A}} d\boldsymbol{x}\, \mu^{(1)}_{N,\sigma^2}(\xi) p^{\xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}) &= \mathbf{E}^{(1)}_{N,\sigma^2}\Big[p^{\Xi}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)})\Big] \\ &= p^{\mu^{(1)}_{N,\sigma^2}}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}), \end{aligned}$$

and that of the RHS of (3.1) gives the multitime joint probability density function of the noncolliding BM starting from $N\delta_0$ with duration $T = \sigma^2$, in which observations are performed at M times in the reversed order $0 < \sigma^2 c_{\sigma^2}(t_M) < \sigma^2 c_{\sigma^2}(t_{M-1}) < \cdots < \sigma^2 c_{\sigma^2}(t_1) < \sigma^2$, multiplied by the scale factors $c_{\sigma^2}(t_m), 1 \leq m \leq M$. Then the equivalence of the processes (2.7) is concluded. In a similar way, we can prove (2.9). ∎

3.2 Proof of Theorem 2.2

For a sequence $(N_m)_{m=1}^M$ of positive integers less than or equal to N, the $(N_1, \dots, N_M)$-multitime correlation function at M times $0 < t_1 < \cdots < t_M < T$ of $(\Xi_T(t), t \in [0,T], \mathbb{P}^{N\delta_0})$ is obtained from (2.1) by

$$
\begin{aligned}
&\rho_T^{N\delta_0}(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}) \\
&= \prod_{m=1}^M \int_{\mathbb{R}^{N-N_m}} \prod_{j=N_m+1}^N \frac{dx_j^{(m)}}{(N-N_m)!} \int_{\mathbb{R}^N} \frac{d\boldsymbol{x}^{(M+1)}}{N!} p_T^{N\delta_0}(t_1, \xi^{(1)}; \dots; t_M, \xi^{(M)}; t_{M+1}, \xi^{(M+1)}).
\end{aligned}
$$

In [38, 15], the functions $\widetilde{\mathrm{A}}_{11}(s,x;t,y;T,t_1), \widetilde{\mathrm{A}}_{12}(s,x;t,y;T,t_1), \widetilde{\mathrm{A}}_{22}(s,x;t,y;T,t_1), 0 < s, t < T, (x,y) \in \mathbb{R}^2$ are given such that

$$
\begin{aligned}
&\rho_T^{N\delta_0}(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}) \\
&\quad = \mathrm{Pf}_{\substack{1\le j \le N_m, 1 \le k \le N_n \\ 1 \le m,n \le M}} \left[\begin{pmatrix} \widetilde{\mathrm{A}}_{11}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; T, t_1) & \widetilde{\mathrm{A}}_{12}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; T, t_1) \\ -\widetilde{\mathrm{A}}_{12}(t_n, x_k^{(n)}; t_m, x_j^{(m)}; T, t_1) & \widetilde{\mathrm{A}}_{22}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; T, t_1) \end{pmatrix} \right].
\end{aligned}
$$

By direct calculation we have found that

$$
\begin{aligned}
&\widetilde{\mathrm{A}}_{jk}(\sigma^2 c_{\sigma^2}(s), c_{\sigma^2}(s)x; \sigma^2 c_{\sigma^2}(t), c_{\sigma^2}(t)y; \sigma^2, c_{\sigma^2}(t_M)) \\
&\quad = c_{\sigma^2}(s)^{-1/2} c_{\sigma^2}(t)^{-1/2} \mathrm{A}_{jk}(s,x;t,y;\sigma^2),
\end{aligned} \tag{3.2}
$$

for $(j,k) = (1,1),(1,2),(2,2)$, where $\mathrm{A}_{jk}, (j,k) = (1,1),(1,2),(2,2)$, are given by (2.11) with (2.12). Note that by definition of Pfaffian (1.10), with any set of factors $v_j, 1 \le j \le 2N$,

$$
\mathrm{Pf}_{1 \le j < k \le 2N}(v_j a_{jk} v_k) = \prod_{j=1}^{2N} v_j \times \mathrm{Pf}_{1 \le j < k \le 2N}(a_{jk}). \tag{3.3}
$$

Then by the equality (2.7) of Lemma 2.1 (see also (3.1)),

$$
\begin{aligned}
&\rho^{\mu_{N,\sigma^2}^{(1)}}(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}) \\
&= \mathrm{Pf}_{\substack{1\le j \le N_m, 1 \le k \le N_n \\ 1 \le m,n \le M}} \left[c_{\sigma^2}(t_m)^{-1} \begin{pmatrix} \mathrm{A}_{11}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2) & \mathrm{A}_{12}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2) \\ -\mathrm{A}_{12}(t_n, x_k^{(n)}; t_m, x_j^{(m)}; \sigma^2) & \mathrm{A}_{22}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2) \end{pmatrix} c_{\sigma^2}(t_n)^{-1} \right] \\
&\qquad \times \prod_{\ell=1}^M c_{\sigma^2}(t_\ell)^N \\
&= \mathrm{Pf}_{\substack{1\le j \le N_m, 1 \le k \le N_n \\ 1 \le m,n \le M}} [\mathbb{A}(t_m, x_j^{(m)}; t_n, x_k^{(n)}; \sigma^2)],
\end{aligned}
$$

and the proof is completed. ∎

3.3 Proof of Theorem 2.3

In [18], the functions $\widetilde{\mathrm{A}}_{11}^{(\nu,\kappa)}(s,x;t,y;T,t_1)$, $\widetilde{\mathrm{A}}_{12}^{(\nu,\kappa)}(s,x;t,y;T,t_1)$, $\widetilde{\mathrm{A}}_{22}^{(\nu,\kappa)}(s,x;t,y;T,t_1)$, $0 < s,t < T, (x,y) \in (0,\infty)^2$ are given such that

$$\begin{aligned}
&\rho_{T,(\nu,\kappa)}^{N\delta_0}(t_1,\boldsymbol{x}_{N_1}^{(1)};\dots;t_M,\boldsymbol{x}_{N_M}^{(M)}) \\
&= \mathrm{Pf}_{\substack{1\le j\le N_m, 1\le k\le N_n \\ 1\le m,n\le M}} \left[\begin{pmatrix} \widetilde{\mathrm{A}}_{11}^{(\nu,\kappa)}(t_m,x_j^{(m)};t_n,x_k^{(n)};T,t_1) & \widetilde{\mathrm{A}}_{12}^{(\nu,\kappa)}(t_m,x_j^{(m)};t_n,x_k^{(n)};T,t_1) \\ -\widetilde{\mathrm{A}}_{12}^{(\nu,\kappa)}(t_n,x_k^{(n)};t_m,x_j^{(m)};T,t_1) & \widetilde{\mathrm{A}}_{22}^{(\nu,\kappa)}(t_m,x_j^{(m)};t_n,x_k^{(n)};T,t_1) \end{pmatrix} \right].
\end{aligned}$$

By direct calculation we have found that

$$\begin{aligned}
&\widetilde{\mathrm{A}}_{jk}^{(\nu,\kappa)}(\sigma^2 c_{\sigma^2}(s), c_{\sigma^2}(s)^2 x; \sigma^2 c_{\sigma^2}(t), c_{\sigma^2}(t)^2 y; \sigma^2, c_{\sigma^2}(t_M)) \\
&\quad = c_{\sigma^2}(s)^{-1} c_{\sigma^2}(t)^{-1} \mathrm{A}_{jk}^{(\nu,\kappa)}(s,x;t,y;\sigma^2), \qquad (3.4)
\end{aligned}$$

for $(j,k) = (1,1),(1,2),(2,2)$, where $\mathrm{A}_{jk}^{(\nu,\kappa)}, (j,k) = (1,1),(1,2),(2,2)$, are given by (2.13) with (2.14). Then by the equality (2.9) of Lemma 2.1 and the property of Pfaffian (3.3), the theorem is proved. ∎

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